AN FDTD ALGORITHM WITH PERFECTLY MATCHED LAYERS FOR CONDUCTIVE MEDIA

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ABSTRACT: We extend Berenger’s perfectly matched layers (PML) to conductive media. A difference-time-domain (FDTD) algorithm with PML as an absorbing boundary condition is developed for solutions of Maxwell’s equations in inhomogeneous, conductive media. For a perfectly matched layer in a conductive medium, an additional term involving the time-integrated electric field has to be introduced to account for the coupling between the loss from the PML and the normal conduction loss. This absorbing boundary condition is proven to be highly effective for the absorption of outgoing waves at the computational edge even when a dipping interface intersects the outer boundary. The algorithm is validated by analytical solutions, and is also compared with Liao’s absorbing boundary condition. Numerical results for subsurface radar measurements are shown to demonstrate the applications of this method. © 1997 John Wiley & Sons, Inc. Microwave Opt Technol Lett 14, 134–137, 1997.

Key words: Finite-difference–time-domain (FDTD) method; perfectly matched layers (PML); conductive medium; inhomogeneous medium; transient wave scattering; subsurface radar

I. INTRODUCTION

Simulations of transient electromagnetic waves in conductive (lossy) media are important in many applications, such as medical imaging, nondestructive evaluation, and geophysical subsurface sensing. So far, the predominant method used for time-domain solutions of Maxwell’s equations is the finite-difference–time-domain (FDTD) method, also known as Yee’s algorithm [1]. The FDTD method in general gives very satisfactory results if the discretization is fine enough.

In the FDTD method for wave-propagation problems in unbounded media, artificial boundary conditions have to be used to eliminate the reflections from the edge of the finite computational domain. These boundary conditions are known as absorbing boundary conditions (ABCs), because they are developed to absorb outgoing waves. Among many different ABCs, the perfectly matched layers (PML) recently introduced by Berenger [2] provide highly effective absorption to the outgoing waves. Because the PML is a material absorbing boundary condition, it is ideal for parallel computation, because unlike most other ABCs, only one set of code is required for the computational domain and for its boundary.

Most of the previous effort on PML applications, however, has been concentrated on nonconductive media [2–4]. For conductive media, the implementation of perfectly matched layers has to be modified to account for the coupling of loss from a PML and that from the regular conduction loss. Here we will adopt the stretching coordinate interpretation of PML [4] and extend the formulation presented for elastic wave propagation presented by Chew and Liu [5–7] to conductive media. The numerical results are validated by analytical solutions, as well as by FDTD results with the use of Liao’s absorbing boundary condition [8].

In Section II, we summarize the formulation of PML for inhomogeneous, conductive media. Numerical results are shown in Section III.

II. FORMULATION

Consider an isotropic, inhomogeneous medium with space-dependent electric permittivity \( \epsilon(r) \), magnetic permeability \( \mu(r) \), and conductivity \( \sigma(r) \). Maxwell’s equations governing electromagnetic fields in the medium are given by

\[
\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} - \mathbf{M},
\]

\[
\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} + \mathbf{J},
\]

\[
\nabla \cdot \epsilon \mathbf{E} = \rho_e,
\]

\[
\nabla \cdot \mu \mathbf{H} = \rho_m,
\]

where \( \mathbf{J} \) and \( \mathbf{M} \) are the imposed electric and magnetic current densities, respectively, and \( \rho_e \) and \( \rho_m \) are the total electric and magnetic charge densities, respectively. The introduction of magnetic sources \( \mathbf{M} \) and \( \rho_m \) is convenient for modeling small loop antennas, which can be approximated as magnetic dipoles.

Because Eqs. (3) and (4) are derivable from (1) and (2), numerical solutions of Maxwell’s equations with the use of the FDTD method for unbounded media require the discretization of Eqs. (1) and (2) and truncation into a finite computation domain. In the traditional Yee’s algorithm, absorbing boundary conditions have to be employed at the computational edge to eliminate the reflections from this artificial boundary.

Recently, Berenger introduced a perfectly matched layer as a material absorbing boundary condition [2]. The PML was alternatively interpreted by Chew and Weeden, who used the stretched coordinates and applied the PML to nonconductive media [4]. For conductive, or lossy, media a slightly more general formulation for PML is adopted, because the new formulation will allow more freedom to include additional attenuation to waves in conductive media. In this work, we will adopt such a more general formulation as given by Chew and Liu [5–7] for elastic wave propagation. The extension of PML to conductive media was also presented in [9, 10]. Fang and Wu present a slightly different formulation for two dimensions [11].

A. Modified Maxwell’s Equations for Conductive Media. In the frequency domain, the modified Maxwell’s equations in stretched coordinates are

\[
\nabla_e \times \mathbf{E} = i \omega \mu \mathbf{H} - \mathbf{M},
\]

\[
\nabla_e \times \mathbf{H} = (-i \omega \epsilon + \sigma) \mathbf{E} + \mathbf{J},
\]

corresponding to the first two equations (1) and (2), where a time dependence of \( e^{-i\omega t} \) is implied. The operator \( \nabla_e \) expressed in terms of the complex coordinate-stretching variables \( e_\eta (\eta = x, y, z) \) is

\[
\nabla_e = \sum_{\eta=x,y,z} \hat{\eta} \frac{1}{e_\eta} \frac{\partial}{\partial \eta}.
\]
For conductive media, the complex coordinate-stretching variable is chosen as

\[ e_q = a_q + i \frac{\omega_q}{\omega} \tag{8} \]

It is noted that the formulation in [4] is a special case of (8), with \( a_q = 1 \). In Eq. (8), the real part \( a_q \) is a scaling factor, and the imaginary part \( \omega_q / \omega \) represents a loss in the PML.

The addition of the real part \( a_q \) in (8) as an independent variable is to accelerate the attenuation to evanescent waves as well as waves in lossy media. This can be easily understood by considering a plane wave \( e^{i k z} \) propagating in the \( x \) direction with a complex wave number \( k = k' + ik'' \). Using \( x = x'(a_q + i \omega_q / \omega) \), we have, in the stretched coordinates

\[ e^{i k z} = \exp \left[ i \left( a_q k' - \frac{\omega_q}{\omega} k'' \right) x' \right] \exp \left[ - \left( a_q k'' + \frac{\omega_q}{\omega} k' \right) x' \right], \tag{9} \]

which implies that the extra variable \( a_q \) provides an additional freedom to further attenuate waves in lossy media. The proof of zero reflections for a PML interface follows that in [4].

B. Splitting of Equations in the Time Domain. Equations (5) and (6) are in the frequency domain. To use the FDTD method, the corresponding time-domain equations can be derived with the use of Eqs. (7) and (8). Following [4], the fields are split as \( E = E^{(s)} + E^{(i)} + E^{(c)} \) and \( \mathbf{H} = H^{(s)} + H^{(i)} + H^{(c)} \). Then, Eqs. (5) and (6) can be converted into the time domain with the split fields (\( \eta = x, y, z \))

\[ a_n \frac{\partial H^{(n)}}{\partial t} + \mu \omega_n H^{(n)} = -\frac{\partial}{\partial \eta} (\hat{n} \times E) - M^{(n)}, \tag{10} \]

\[ a_n \epsilon \frac{\partial E^{(n)}}{\partial t} + (\epsilon a_n \sigma + \omega_n \epsilon) E^{(n)} + \omega_n \sigma \int_{-\infty}^{t} E^{(q)} \, dt = \frac{\partial}{\partial \eta} (\hat{n} \times \mathbf{H}) - J^{(n)}. \tag{11} \]

Equations (10) and (11) consist of a total of 12 scalar equations, because both \( E^{(n)} \) and \( H^{(n)} \) have two scalar components perpendicular to \( \hat{n} \). In the above, the source terms are also split, so that

\[ J = \sum_{\eta=x,y,z} J^{(n)}, \quad M = \sum_{\eta=x,y,z} M^{(n)}. \tag{12} \]

The split equations (10) and (11) can then be solved by the following FDTD method. Note that with the introduction of PML, there is an additional term involving the time-integrated electric field in Eq. (11). This term represents the coupling of the loss in PML with the regular conduction loss.

C. Discretization. We adopt Yee’s algorithm to discretize the split equations (10) and (11). If the finite-difference cell sizes are \( \Delta x, \Delta y, \) and \( \Delta z \) in the \( x, y, \) and \( z \) directions, respectively, the field components for the \( \frac{1}{2} \) cell is shown in Figure 1. It should be noted that the time-integrated electric field

\[ \mathbf{E}^{(n)} = \int_x \mathbf{E}^{(n)} \, dt \text{ is located at the same position in the staggered grid as the electric field } \mathbf{E}^{(n)}. \]

As in the standard Yee's algorithm, central differencing is used for both spatial and temporal derivatives. Furthermore, the second and third terms on the left-hand side of Eq. (11) require the averaging of their values at \( t = n \Delta t \) and \( t = (n + 1/2) \Delta t \), because \( \mathbf{E} \) is not evaluated at \( t = (n + 1/2) \Delta t \). This averaging has the same second-order accuracy as the central differencing used for the first term on the left-hand side of (11), and therefore does not degrade the overall accuracy in the discretization.

As an example, the time-stepping equation for the \( x \) component of \( \mathbf{E}^{(n)} \) in Eq. (11) after discretization is given by

\[ E^{(n)}_x(i, j, k; n + 1) = f_1 E^{(n)}_x(i, j, k; n) + f_2 [H_y(i, j, k; n) - H_y(i, j - 1, k; n)] + f_3 E^{(n)}_x(i, j, k; n) \frac{\Delta t}{\epsilon + \sigma \Delta t/2} \times J^{(n)}_y(i, j, k; n + \frac{1}{2}), \tag{13} \]

where \((i, j, k)\) are the spatial indices, and \( n \) is the time index. The coefficients \( f_1, f_2, \) and \( f_3 \) in (13) are given by

\[ f_1 = \frac{\epsilon \left( a_y \frac{\Delta y}{\Delta t} - \frac{\omega_y}{2} \right) - a_y \sigma \frac{1}{2}}{\epsilon \left( a_y \frac{\Delta y}{\Delta t} + \frac{\omega_y}{2} \right) + \frac{\sigma}{2} (a_y + \omega_y \Delta t)}, \tag{14} \]

\[ f_2 = \frac{1/\Delta y}{\epsilon \left( a_y \frac{\Delta y}{\Delta t} + \frac{\omega_y}{2} \right) + \frac{\sigma}{2} (a_y + \omega_y \Delta t)}, \tag{15} \]

\[ f_3 = -\frac{\omega_y \sigma \Delta t}{\epsilon \left( a_y \frac{\Delta y}{\Delta t} + \frac{\omega_y}{2} \right) + \frac{\sigma}{2} (a_y + \omega_y \Delta t)}. \tag{16} \]

Similar expressions can be derived for the other field components.
III. NUMERICAL RESULTS

We have implemented the FDTD algorithm with the perfectly matched layers as the absorbing boundary condition. Because of the discretization, some reflection will occur at a PML interface. This reflection is proportional to the contrast in the coordinate stretching variables. Therefore, to minimize the reflection from the PML layers, we choose a linear profile for the PML coordinate-stretching variables. Typically we use 10 layers of perfectly matched layers at the computational edge.

The algorithm has been implemented for one, two, and three dimensions. Numerical results are shown below to demonstrate the applications of the FDTD algorithm. In the following examples, an electric dipole directed in the $\hat{x}$ direction is used as a source, and the field component $E_x$ is measured at a series of receiver locations. The time function of the source is the first derivative of the Blackman–Harris window function, and is given by

$$f(t) = \begin{cases} \frac{n=1}{n=1} \frac{n\pi}{T} a_n \sin(2n\pi t/T), & \text{for } 0 < t < T, \\ 0, & \text{otherwise}, \end{cases}$$

(17)

where $T$ is the duration of the source function, and the coefficients are $a_1 = -0.488$, $a_2 = 0.145$, $a_3 = -0.0102222$. The central frequency of this function is defined as $f_c = 1.55/T$. In the following examples, this central frequency is chosen to be $f_c = 10\text{ MHz}$.

A. A Homogeneous Conductive Medium. The first simple test-case for the FDTD algorithm is a homogeneous medium with $\varepsilon_r = 1$, $\mu_r = 1$, $\sigma = 10^{-4}$ S/m. The electric dipole source is located at the origin, and the $E_x$ components are calculated at 10 locations $x_i = 4 + (i - 1) \times 0.5$; $(y_i, z_i) = (0, 2)\text{ m}$ $(i = 1, \ldots, 10)$. The FDTD results are compared with the analytical solutions in Figure 2(a) for the fourth receiver, and in Figure 2(b) for the array. Note that the results are normalized with respect to the peak value of the field at the fourth receiver.

B. A Conductive Sphere in a Conductive Background Medium. Another special case where analytical solutions are available is a dipole source in a conductive sphere. We study the electromagnetic field due to an electric dipole source (polarized in the $x$ direction) at the center $(x, y, z) = (0, 0, 0)$ of a conductive sphere with $\varepsilon_r = 1.5; \mu_r = 1; \sigma = 5 \times 10^{-5}$ S/m. The sphere is surrounded by a background conductive medium with $\varepsilon_r = 4; \mu_r = 1; \sigma = 2 \times 10^{-4}$ S/m. The $E_x$ component are then calculated at 10 locations $x_i = 4 + (i - 1) \times 0.5$; $(y_i, z_i) = (0, 2)\text{ m}$ $(i = 1, \ldots, 10)$. Again, Figures 3(a) and 3(b) show the comparison of the FDTD results with the analytical solutions. Excellent agreement is observed.

C. Comparison with Liao’s ABC. One important application of this algorithm is in simulations of subsurface radar measurements. We compare the FDTD results with PMLs and those using Liao’s absorbing boundary condition [8] for a subsurface radar measurement of objects in a three-layer medium. The three (from top to bottom) layers have the following properties: $\varepsilon_r = 1; \mu_r = 1; \sigma = 0; \varepsilon_r = 2; \mu_r = 1; \sigma = 5 \times 10^{-5}$ S/m; $\varepsilon_r = 4; \mu_r = 1; \sigma = 2 \times 10^{-4}$ S/m. These layers extend to the full range in $y$ direction. Two rectangular cylinders are buried in the second layer and extend from $y = -5$ to $y = 5$. The $x$-$z$ cross section at $y = 0$ is shown in Figure 4(a). An electric dipole source is located at the origin, and an array of 44 receivers is located at the ground surface (between the first and second layers) $x_i = (i - 1) \times 0.5 - 11; y_i = 0; z_i = -2.5\text{ m}$ $(i = 1, \ldots, 44)$.

Because in this problem the material properties near all outer boundaries (within seven grids) do not change in the directions perpendicular to the boundaries, Liao’s absorbing boundary condition can also be applied. However, Liao’s ABC requires double precision in order to be stable, whereas the PMLs require only single precision. We compare the FDTD results using PMLs and those using Liao’s ABC, as shown in Figure 4(b) for the tenth receiver. Note that these waveforms are the total $E_x$ component. The small scattered

![Figure 2](image1.png)  
**Figure 2** Comparison of FDTD results with analytical solutions for a homogeneous, conductive medium. (a) The $E_x$ component at the fourth receiver. (b) The $E_x$ component at the array of receivers. The field is normalized with respect to the peak value at the fourth receiver.

![Figure 3](image2.png)  
**Figure 3** FDTD results and analytical solutions for a conductive sphere in a background conductive medium. (a) The $E_x$ component at the fourth receiver. (b) The $E_x$ component at the array of receivers. The field is normalized with respect to the peak value at the fourth receiver.
field is buried in the large direct arrivals. Figure 4(c) shows the scattered field by subtracting the direct field in the three-layer medium (without the two square cylinders) from the total field.

D. A Dipping Interface in Surface Radar Measurements. Dipping interfaces are often encountered and are therefore very important for subsurface radar measurements. Unfortunately, previous absorbing boundary conditions cannot model these dipping interfaces, because all of them require at least homogeneity in the direction perpendicular to the truncating boundary. The PML absorbing boundary condition therefore offers an unparalleled advantage over other ABCs for these problems.

We illustrate such applications of PMLs for a dipping layer shown in Figure 5(a). The configuration for the three layers is the same as that in Figure 4(a), except that the bottom layer is dipping. In addition to two rectangular cylinders, a metallic sphere of radius 4 m is also buried in the second layer. For this problem with a dipping interface, the FDTD algorithm with PMLs is stable, whereas with Liao’s ABC, the algorithm becomes unstable as soon as the waves propagate to the boundary. Figure 5(b) shows the total field measured at the 44 receivers along the ground surface. By subtracting the direct arrivals due to the three-layer medium, the scattered field is clearly shown in Figure 5(c).

IV. CONCLUSIONS

We have developed an FDTD algorithm with the perfectly matched layers as an absorbing boundary condition for inhomogeneous, conductive media. We have validated the FDTD algorithm with Liao’s absorbing boundary condition, as well as analytical solutions for a homogeneous medium and for a conductive sphere. Unlike the previous ABCs, the PML absorbing boundary condition is stable even when a dipping interface intersects the truncating boundary. This algorithm is ideal for parallel computation because the same code is used for inner computation domain and for the outer boundary.

REFERENCES