
© 1998 John Wiley & Sons, Inc.
CCC 0895-2477/98

QUASI-PML FOR WAVES IN CYLINDRICAL COORDINATES
Q. H. Liu1 and J. Q. He1
1Kiplach School of Electrical and Computer Engineering
New Mexico State University
Las Cruces, New Mexico 88003

Received 3 March 1998

ABSTRACT: We prove that the straightforward extension of Berenger’s original perfectly matched layer (PML) is not reflectionless at a cylindrical interface in the continuum limit. A quasi-PML is developed as an absorbing boundary condition (ABC) for the finite-difference time-domain method in cylindrical coordinates. For three-dimensional problems, this quasi-PML requires only ten equations, instead of 12 equations in the best true PML formulations. With a satisfactory absorption level, it is simpler, and requires about 20% less computer memory than the true PML ABC. © 1998 John Wiley & Sons, Inc. Microwave Opt Technol Lett 19: 107–111, 1998.

Key words: absorbing boundary condition; quasi-PML; electromagnetic wave propagation

I. INTRODUCTION
Since Berenger introduced the perfectly matched layer (PML) as an absorbing boundary condition [1], many researchers have extended the theory and applied this ABC to various problems. In particular, Berenger [2] and Katz, Thiele, and Taflove [3] extend the PML to three dimensions. Chew and Weedon propose an elegant reformulation of PML using the stretched coordinates [4]. So far, however, most work on PML has been for waves in Cartesian coordinates. It is shown rigorously that the reflection coefficient at a planar interface between a normal material and a PML medium can be made identical zero in the continuous limit [4].

Although there are studies on PML for nonorthogonal grids, notably [5–8], little theoretical analysis has been done to prove the reflectionless PML (or on the contrary) for general curvilinear interfaces. Only recently have the quasi-PML and true PML ABCs been implemented in cylindrical coordinates [9–13].

This work first proves that the straightforward extension of the original PML is no longer reflectionless at a cylindrical interface at the continuum limit. Therefore, this extension is called the quasi-PML. For most applications where the outer boundary is a few wavelengths away from the origin, this quasi-PML provides high absorption, while the implementation is simple. For three-dimensional problems, it requires only ten equations. In contrast, all of the known implementations of the true PML [10–13] require at least 12 equations, demanding roughly 20% more computer memory. Finite-difference results are shown to support our analysis.

II. FORMULATION
A. Stretched Cylindrical Coordinates. Consider vector electromagnetic waves in a homogeneous medium. Similar to Cartesian coordinates [4], we define stretched cylindrical coordinates (r, θ, z) such that the original curl and divergence operators become

\[
\nabla \times \mathbf{E} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E_\theta}{\partial \theta} - \frac{\partial E_z}{\partial z} \right) + \hat{\theta} \left( \frac{1}{\varepsilon_r} \frac{\partial E_\theta}{\partial r} - \frac{1}{\varepsilon_z} \frac{\partial E_z}{\partial r} \right) \\
\n\nabla \cdot \mathbf{E} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial (rE_\theta)}{\partial r} \right) + \frac{1}{\varepsilon_r} \frac{\partial E_\theta}{\partial r} + \frac{1}{\varepsilon_z} \frac{\partial E_z}{\partial z} 
\]

where (ε_r, ε_\theta, ε_z) are the complex stretching variables in the (r, \theta, z)-directions. It is important to note that even in the \theta-direction, a stretching variable ε_\theta is introduced for consistency, as discussed below. Maxwell’s equations in these stretched coordinates can then be used to derive the following Helmholtz equations for a homogeneous, source-free region with constant μ and ε:

\[
(\nabla^2 + k^2) \begin{pmatrix} E_r \\ H_\theta \\ H_z \end{pmatrix} = 0 
\]

where \( k^2 = \omega^2 \mu \varepsilon \), and

\[
\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{\varepsilon_r} \frac{\partial \varepsilon_r}{\partial \theta} \frac{\partial}{\partial \theta} + \frac{1}{\varepsilon_z} \frac{\partial}{\partial z} \frac{\partial}{\partial z}. 
\]

In general, the eigensolutions to Eq. (3) should be written as combinations of cylindrical harmonics. In particular, we write the solution as

\[
\begin{pmatrix} E_r \\ H_\theta \\ H_z \end{pmatrix} = \begin{pmatrix} E_0 \\ H_\theta \end{pmatrix} H^{(1)}_n(k,r) e^{i(k_z z + n \theta)}
\]

where \( H^{(1)}_n \) is the nth-order Hankel function of first kind. In order for Eq. (5) to be the eigensolution of Eq. (3), it is required that

\[
k^2 - \frac{k_r^2}{\varepsilon_r} - \frac{k_\theta^2}{\varepsilon_\theta} + \frac{n^2}{r^2} \left( \frac{1}{\varepsilon_r} - \frac{1}{\varepsilon_\theta} \right) = 0
\]

for all r. This implies two conditions: 1) the dispersion relation \( k^2 = k_r^2/\varepsilon_r + k_\theta^2/\varepsilon_\theta \), and 2) \( \varepsilon_\theta = \varepsilon_r \). The second condition is simply a consequence of the eigensolution.
Therefore, \( e_p = e_r \) is necessary for consistency in cylindrical coordinates.

Now, consider the behavior of electromagnetic waves in a PML medium in cylindrical coordinates. If \( e_{r_{c}} = 1 \) and \( e_r = a_r + i a_i / a_0 \), the radial wavenumber \( k_r \) obtained from the dispersion relation will have a positive imaginary part, which ensures that \( H^{(1)}(k_r, r) \), and thus the field, decays rapidly in the radial direction.

**B. Reflections from a Cylindrical Interface.** With the above derivation for the quasi-PML medium in stretched cylindrical coordinates, we now study the reflection at a cylindrical interface at \( r = a \) between a normal medium and an outer PML medium. Similar to Cartesian coordinates, we choose the normal medium with \( (\mu_n, e_n) = (\mu, e) \) and \( (\epsilon_n, \epsilon_{r_{c}}, \epsilon_{z}) = (1, 1, 1) \), and the PML medium with \( (\mu_s, e_{s}) = (\mu, e) \) and \( (\epsilon_{s2}, \epsilon_{s2}, \epsilon_{s1}) = (e_s, e_s, 1) \), where \( e_r = a_r + i a_i / a_0 \neq 1 \).

From the phase matching, \( k_{s1} = k_{s2} = k_s \). According to the dispersion relations, \( k_{n1} = \sqrt{k^2 - k_{s1}^2} \) and \( k_{n2} = e_s k_{s1} \neq k_{s1} \). Similar to the case where both media are normal, we write the field inside and outside the interface as

\[
\begin{bmatrix}
E_z \\
H_z
\end{bmatrix} = \begin{bmatrix}
H^{(1)}(k_{s1}, r) a_1 + J_s(k_{s1}, r) \mathbf{R} \cdot a_1 \\
J^{(1)}(e_s k_{s1}, r) \mathbf{T} \cdot a_1
\end{bmatrix}
\]

for \( r < a \)

\[
\begin{bmatrix}
E_z \\
H_z
\end{bmatrix} = \begin{bmatrix}
H^{(1)}(e_s k_{s1}, r) \mathbf{R} \cdot a_1 \\
H^{(1)}(e_s k_{s1}, r) \mathbf{T} \cdot a_1
\end{bmatrix}
\]

for \( r > a \) (7)

where \( a_1 \) is the source excitation vector, and \( \mathbf{R} \) and \( \mathbf{T} \) are the reflection and transmission matrices of dimension \( 2 \times 2 \). Note that, except for \( n = 0 \), \( \mathbf{R} \) and \( \mathbf{T} \) are not diagonal matrices, indicating the coupling between TE and TM waves at a cylindrical interface. From these longitudinal components, one can find the transverse field components. Then, by matching the boundary conditions across the interface at \( r = a \), one arrives at the reflection matrix

\[
\mathbf{R} = \begin{bmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{bmatrix}
= \mathbf{\bar{D}}^{-1} \mathbf{N}_R \mathbf{\bar{D}}^{-1} \begin{bmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22}
\end{bmatrix}
\]

(8)

where matrix \( \mathbf{N}_R \) is given by

\[
\begin{align*}
N_{11} &= -i \omega x_1 [J^{(1)}(e_{x_1}) H^{(1)}(e_{x_1}) - H^{(1)}(e_{x_1}) J^{(1)}(e_{x_1})] \\
N_{12} &= N_{21} = -nk_z \left( 1 - \frac{1}{e_r} \right) \frac{1}{e_r} [J^{(1)}(e_{x_1}) H^{(1)}(e_{x_1})] \\
N_{22} &= i \omega x_1 [J^{(1)}(e_{x_1}) H^{(1)}(e_{x_1}) - H^{(1)}(e_{x_1}) J^{(1)}(e_{x_1})]
\end{align*}
\]

and matrix \( \mathbf{\bar{D}} \) is given by

\[
\begin{bmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}
\]

\[
\begin{align*}
D_{11} &= i \omega x_1 [J^{(1)}(e_{x_1}) H^{(1)}(e_{x_1}) - H^{(1)}(e_{x_1}) J^{(1)}(e_{x_1})] \\
D_{12} &= D_{21} = nk_z \left( 1 - \frac{1}{e_r} \right) J^{(1)}(e_{x_1}) H^{(1)}(e_{x_1}) \\
D_{22} &= -i \omega x_1 [J^{(1)}(e_{x_1}) H^{(1)}(e_{x_1}) - H^{(1)}(e_{x_1}) J^{(1)}(e_{x_1})]
\end{align*}
\]

(10)

and \( x_1 = k_{s1} a \). The prime in Eqs. (9) and (10) denotes the derivative of a function with respect to its argument.

From Eqs. (8) and (9), it is observed that \( \mathbf{R} \) is nonzero in general since \( e_r \neq 1 \). For \( n \neq 0 \), i.e., the nonaxisymmetric case, the off-diagonal elements are not equal to zero, giving rise to coupling between TE and TM waves at the PML interface. For the axisymmetric case (\( n = 0 \)), the off-diagonal elements are zero, and it is possible to make the reflection matrix zero, but only at certain frequencies. Therefore, the cylindrical PML interface is not reflectionless for all frequencies and all incidence angles.

A closer investigation of Eqs. (8) and (9) leads to the following conclusions: 1) the reflection decreases with an increase in frequency, as \( \mathbf{R} \) approaches zero as \( f \rightarrow \infty \) by asymptotic expressions of Hankel functions; 2) the reflection increases as the incidence angle increases (i.e., \( \theta \) increases) and as the azimuthal order \( n \) increases; and 3) the TE/TM coupling is roughly proportional to the contrast in \( 1/e_r \), as indicated by \( N_{12} \) and \( N_{21} \).

A similar formulation can be obtained for scalar waves. For example, the scalar reflection coefficient for acoustic waves at a cylindrical interface between a normal material and a PML medium is given by

\[
R = \frac{H^{(1)}(e_1, \theta) H^{(1)}(e_1, \phi)}{J^{(1)}(e_1, \theta)} - \frac{H^{(1)}(e_1, \phi) J^{(1)}(e_1, \phi)}{J^{(1)}(e_1, \theta)}
\]

which approaches zeros also only at high frequencies.

**C. Quasi-PML ABC for Cylindrical Coordinates.** Using the stretched coordinates in (1) and (2), one can easily derive the split Maxwell’s equations [4]. For an inhomogeneous conductive medium, following the procedures in [14, 15], we split the field component \( E_r = E_{r1}^{(d)} + E_{r2}^{(s)} \) and similarly for the other field components \( E_{\theta}, H_r, \) and \( H_\phi \). Then, Ampère’s law can be split as

\[
a_r \frac{\partial E_{r1}^{(d)}}{\partial t} + (\omega_r + a, \sigma) E_{r2}^{(s)} + \omega_r \sigma \int_{-\infty}^{t} E_{r1}^{(s)}(\tau) d\tau
\]

\[
= \frac{1}{r} \frac{\partial H_\phi}{\partial \theta} - J_{r1}^{(s)}
\]

(11a)

\[
a_r \frac{\partial E_{r2}^{(s)}}{\partial t} + (\omega_r + a, \sigma) E_{r2}^{(s)} + \omega_r \sigma \int_{-\infty}^{t} E_{r1}^{(s)}(\tau) d\tau
\]

\[
= - \frac{\partial H_\phi}{\partial z} - J_{r2}^{(s)}
\]

(11b)

\[
a_r \frac{\partial E_{\theta1}^{(d)}}{\partial t} + (\omega_r + a, \sigma) E_{r2}^{(s)} + \omega_r \sigma \int_{-\infty}^{t} E_{\theta1}^{(s)}(\tau) d\tau
\]

\[
= - \frac{\partial H_{r1}}{\partial r} - J_{\theta1}^{(s)}
\]

(11c)

\[
a_r \frac{\partial E_{\theta2}^{(s)}}{\partial t} + (\omega_r + a, \sigma) E_{r2}^{(s)} + \omega_r \sigma \int_{-\infty}^{t} E_{\theta1}^{(s)}(\tau) d\tau
\]

\[
= \frac{\partial H_{r1}}{\partial z} - J_{\theta2}^{(s)}
\]

(11d)

\[
a_r \frac{\partial E_z^{(s)}}{\partial t} + (\omega_r + a, \sigma) E_z + \omega_r \sigma \int_{-\infty}^{t} E_z(\tau) d\tau
\]

\[
= \frac{1}{r} \frac{\partial H_\phi}{\partial r} - \frac{1}{r} \frac{\partial E_r}{\partial \theta} - J_z
\]

(11e)
Similarly, Faraday’s law can be split as

\begin{align}
a_r \mu \frac{\partial H_r^{(z)}}{\partial t} + \omega_r \mu H_r^{(z)} &= \frac{1}{r} \frac{\partial E_z}{\partial \theta} + M_r^{(z)} \\
a_r \mu \frac{\partial H_{\theta}^{(z)}}{\partial t} + \omega_r \mu H_{\theta}^{(z)} &= \frac{\partial E_r}{\partial z} + M_{\theta}^{(z)} \\
a_r \mu \frac{\partial H_{\phi}^{(z)}}{\partial t} + \omega_r \mu H_{\phi}^{(z)} &= -\frac{\partial E_z}{\partial r} + M_{\phi}^{(z)} \\
a_r \mu \frac{\partial H_z^{(z)}}{\partial t} + \omega_r \mu H_z &= -\frac{1}{r} \frac{\partial (rE_{\theta})}{\partial r} + \frac{1}{r} \frac{\partial E_{\phi}}{\partial \theta} + M_z 
\end{align}

Note that because \( e_r = e_{\theta} \), there is no need to split \( E_r \) and \( H_z \) in Eqs. (12e) and (13e). Therefore, there are only ten
equations in the 3-D quasi-PML formulations. In contrast, the minimum number of equations in the true PML formulations for three dimensions is 12, requiring 20% more memory than the quasi-PML [10–13]. Although it can be shown theoretically that this PML is not perfectly matched (thus the name quasi-PML), practically, it provides a satisfactory ABC because the outer boundary is often at least several wavelengths away from the origin. This has been confirmed by many numerical examples including those presented below.

III. NUMERICAL RESULTS

Figure 1 shows this reflection coefficient in (11) for scalar waves as a function of frequency for \( n = 0, 5, \) and 10. The incidence angle is 45°, and \( e_r = 1 + i/\omega \) (frequency is 1 kHz). The reflection coefficient decreases with \( ka \) (the normalized radius of the PML interface), and increases with \( n \), confirming our theoretical prediction. Similar plots have been obtained for the vector electromagnetic case.

Since the above cylindrical PML interface is no longer reflectionless, we refer to this type of boundary condition as a quasi-PML ABC. We have implemented the quasi-PML ABC in the FDTD methods for 2-D polar \((r, \theta)\) and 2-D axisymmetric \((r, z)\) problems, as well as the full 3-D \((r, \theta, z)\) problems using Yee’s algorithm.

We first study a magnetic line \((M_r)\) source in vacuum. This problem can be solved by using 2-D polar coordinates. The problem is discretized into \( N_r \times N_\theta = 128 \times 128 \) cells with \( \Delta r = 5 \) cm. The source has the first derivative of the Blackman Harris window time function with a center frequency of 300 MHz. The source is located at the cell \((35, 70)\), while a receiver is located at cell \((33, 70)\). Figure 2 shows the snapshots of \( H_r \) at time steps \( n = 1000 \) where \( j = 1, \ldots, 15 \) and \( \Delta t = 2.5 \) ps. No visible reflections can be observed at the late time. Figure 3 confirms the excellent agreement between the numerical result and analytical solution. The reflection from the quasi-PML ABC is only 0.45% of the direct field.

Shown in Figure 4 is the \( H_z \) component due to a z-directed magnetic dipole in a homogeneous conductive medium with \( \varepsilon_r = 1, \mu_r = 1, \sigma = 10^{-4} \) S/m. The center frequency of the source is 100 MHz. (a) Array \( H_z \) waveform at the axis. (b) Waveform at the sixth receiver.

IV. CONCLUSIONS

Closed-form solutions have been obtained for the reflection matrix and reflection coefficient for vector and scalar waves at a cylindrical interface between a normal material and a quasi-PML medium. The amount of reflection is a function of frequency, incidence angle, azimuthal variation, and the contrast in PML parameters. At the high-frequency limit, the reflection approaches zero since the curvature of the interface becomes negligible. Numerical results from 2-D and 3-D FDTD methods indicate that this quasi-PML ABC can provide satisfactory absorption to the outgoing waves. Compared to true PML ABCs, this quasi-PML requires only ten field variables instead of at least 12, saving about 20% computer memory. Similar results are expected for spherical coordinates.

ACKNOWLEDGMENT

This work was supported by the U.S. Environmental Protection Agency under PECASE Grant CR-825-225-010, and by the National Science Foundation under CAREER Grant ECS-9702195.

REFERENCES

4. W. C. Chew and W. H. Weedon, “A 3D Perfectly Matched Medium from Modified Maxwell’s Equations with Stretched Co-
DESIGN CONSIDERATIONS FOR RECTANGULAR MICROSTRIP ANTENNA ELEMENTS WITH VARIOUS SUBSTRATE THICKNESSES

Mehmet Kara1

1Weapons Systems Division
Defence Science and Technology Organisation
Salisbury, SA 5108, Australia

Received 25 February 1998

ABSTRACT: The effective design and analysis of microstrip antennas presupposes the quantitative knowledge of the effects of the physical and mechanical properties of the patch, the ground plane, and the substrate material on electromagnetic properties of the antenna. The behavior of antenna electromagnetic properties for varying physical and mechanical properties of the patch, the ground plane, and the substrate material is presented. A set of graphs for a relevant set of parameters is presented which enables the designer to clearly see the tradeoffs involved in choosing specific parameters for the antenna. Several parameters that are deterrents to the proper functioning of the antennas are also identified.